A Causal Construction of Diffusion Processes

Tadeusz BANEK
Dept. of Quantitative Method, Faculty of Management
Tech. Univ. of Lublin, Nadbystrzycka 38, 20-618 Lublin, Poland,
e-mail; t.banek@pollub.pl

January 15, 2010

Abstract

A simple nonlinear integral equation for Ito's map is obtained. Although, it does not include stochastic integrals, it does give causal construction of diffusion processes which can be easily implemented by iteration systems. Applications in financial modelling and extension to fBm are discussed.

key words: diffusion processes, fBm, translation of Wiener processes, Girsanov theorem

1 Introduction

Diffusions are an important class of stochastic processes. They are Markov, and have continuous trajectories. There are extensive, competent historical surveys of the topic by D.W. Stroock given in [3, 5, 6], and we recommend Stroock's discussion to the interested reader. Here, we shall point out only a main stages. Historically, the first construction was given by A.N. Kolmogorov in his 1931 famous paper [1], and, since then, a problem of constructing diffusions in $\mathbb{R}^n$, having differential operators as generators, and no barriers, is known as the Kolmogorov problem. We shall restrict our attention to the later class and call it (for short), $K-$ diffusions. The second construction of $K-$ diffusions was given by K. Ito in [2] (and it is called Ito diffusions too). The theory of ordinary-stochastic equations by H. Sussman [7] and H. Doss [8], and some modification (see [10]), may be regarded as a deterministic variant of Ito's theory. The third is known as a solution of D.W. Stroock and S.R.S. Varadhan martingale problem (see [3]). The fourth is given by the Isobe-Sato formula [4], which gives Wiener-Ito integrals for chaos decomposition of $K-$ diffusions.

In this paper, we propose a new, pathwise variant of Ito's construction of $K-$ diffusions. Although the construction uses a Wiener process (Ito's idea), it does not involve Ito's integrals. It consists in:

(a) Solving a nonlinear, deterministic, Volterra type integral equation

$$c\left(w\left(t\right)-\int_{o}^{t}\varkappa\left(x\left(s\right)\right)ds\right)=x\left(t\right) \tag{1}$$

where $w\in C_{T}\triangleq C\left(\left[0,T\right];\mathbb{R}\right)$, c and $\varkappa$ are ordinary scalar function to be specify later. Under mild assumptions (1) can be solved pathwise and nonanticipative, i.e., for any w, $v\in C_{T}$ one finds x_{w}, $x_{v}\in C_{T}$, such that restrictions x_{w}, x_{v}, on $\left[0,t\right]$ coincide, if $w\left(s\right)=v\left(s\right)$, $s\in\left[0,t\right]$.

(b) Forming a map $X_{t}\left(w\right):\left[0,T\right]\times C_{T}\rightarrow\mathbb{R}$, such that $X_{t}\left(w\right)=x_{w}\left(t\right)$. Hence, $X\left(w\right)$ belongs to the space $\mathfrak{G}\left(C_{T}\right)$ of all nonanticipative mappings from C_{T} to C_{T}, and it is a fix point of the operator $\mathfrak{L}:\mathfrak{G}\left(C_{T}\right)\rightarrow\mathfrak{G}\left(C_{T}\right)$, defined by

$$\mathfrak{L}\left(X\left(w\right)\right)\left(t\right)=c\left(w\left(t\right)-\int_{o}^{t}\varkappa\left(X\left(w\right)\left(s\right)\right)ds\right) \tag{2}$$

where we adopt the convention $X\left(w\right)\left(t\right)=X_{t}\left(w\right)$.

(c) Showing that $X_{t}\left(w\right)$ is a K$-$ diffusions assuming that $w\left(t\right)$, $t\in\left[0,T\right]$ is a Wiener process.

(d) Proving, it is true in the opposite direction as well, i.e., if $X_{t}\left(w\right)$ is a K$-$ diffusion, then it is a fix point of $\mathfrak{L}$.

It is instructive, to compare an intuitive picture behind Ito's theory (here, we again recommend Stroock [3, 5, 6]), with the picture of K$-$ diffusions as suggested by (1). In the first picture, infinitesimal increments of K$-$ diffusions are resulting from combined effects of two forces: a deterministic drift and random (Gaussian) fluctuations. Since combination here means a sum, the both forces (deterministic and random) have the same status in creation of K$-$ diffusions. However, (1) suggests other picture, or looking from cybernetic perspective, better to say a "behavior". Namely, x_{w} follows w, what is easily visible on the diagram below

$$\left[\begin{array}{ccc} w\longrightarrow^{(+)}\otimes & \longrightarrow\left[c\left(\cdot\right)\right]\longrightarrow\downarrow\longrightarrow & x_{w}\\ {}^{(-)}\uparrow\longleftarrow\left[\int\right] & \longleftarrow\left[\varkappa\left(\cdot\right)\right] & \end{array}\right]$$

which explain the idea of simple iteration system which works according to (1). With $y\left(t\right)\triangleq\int_{0}^{t}\varkappa\left(x\left(s\right)\right)ds$, this behavior is even more explicit

$$\frac{d}{dt}y\left(t\right)=\varkappa\circ c\left(w\left(t\right)-y\left(t\right)\right)$$

Hence $y_{w}\left(t\right)\triangleq\int_{0}^{t}\varkappa\left(x_{w}\left(s\right)\right)ds$ follows w with the speed equals the image of the difference $w\left(t\right)-y_{w}\left(t\right)$ under $\varkappa\circ c$. Thus, in this picture we have pure **deterministic** mechanism, expressed in the terms of $\varkappa\circ c$ composition, which forces y_{w} to follow a **random** path w. Even more, a rule of producing actions according to the current errors is known in Automatic Control as a classical **feedback rule**, which in turns, is the most transparent idea of **Cybernetics**. Is there any Variational Principle responsible for this rule is an open question.

The paper is organized as follows. In a preliminary section we state an auxiliary result on (1). In the next section we prove an equivalence theorem, which is the main result of this paper. Several corollaries are also included. Indication for financial mathematics is discussed next. Finally, a partial extension to fBm is included in the last section.

2 Preliminaries

We state here the following

Lemma 1 *Assume $c:\mathbb{R}\to\mathbb{R}$ is locally Lipschitz and $\varkappa:\mathbb{R}\to\mathbb{R}$ measurable and bounded. Then, (a) for any $w\in C_T$, there exists a unique $x_w\in C_T$ satisfying (1), (b) for any $w\in C_T$ and any $\xi\in C_T$, a sequence of successive approximation*

$$\begin{aligned} x_0 &= \xi,\;\; x_{n+1}=\Phi_w(x_n) \\ \Phi_w(x)(t) &\triangleq c\left(w(t)-\int_o^t \varkappa(x(s))\,ds\right) \end{aligned}$$

is convergent in any norm $\|\cdot\|_\lambda$, $\lambda\geq 0$, to x_w, where $\|x\|_\lambda=\max\left\{e^{-\lambda t}|x(t)|;0\leq t\leq T\right\}$, (c) a mapping $C_T\ni w\mapsto X(w)\triangleq x_w\in C_T$ is locally Lipschitz (in any $\|\cdot\|_\lambda$, $\lambda\geq 0$), and nonanticipating.

Proof. The proof consists in two steps. In the first, one can show (a),(b),(c) hold when c is globally Lipschitz. In the second, one can apply a method of continuation in the locally Lipschitz case. The proof is standard hence it is omitted. ■

3 Equivalence theorem

Let $g\in C^1(\Delta)$, $\Delta\subset\mathbb{R}$ and $f\in C(\mathbb{R})$. Define two functions:

$$c'(x)=g(c(x)) \tag{3}$$

and

$$\varkappa(x)=\frac{g'(x)}{2}-\frac{f(x)}{g(x)} \tag{4}$$

Example 2 *(a) Let $g(x)=\sqrt{1+x^2}$. Then $c(x)=\sinh(a+x)$. For an arbitrary $\phi\in C(\mathbb{R})$, set $f(x)=\frac{x}{2}-\phi(x)\sqrt{1+x^2}$, then $\varkappa(x)=\phi(x)$. (b) Let $g(x)=|x|^\alpha, |\alpha|<1$. Then for $a\in\mathbb{R}$, we have $c(x)=[sign(a+x)]\left[(1-\alpha)|a+x|\right]^{1/1-\alpha}$. For $\phi\in C(\mathbb{R})$, set $f(x)=\frac{\alpha}{2}sign(x)|x|^{2\alpha-1}-|x|^\alpha\phi(x)$, then $\varkappa(x)=\phi(x)$.*

Theorem 3 *Assume $c:\mathbb{R}\to\mathbb{R}$, $\varkappa:\mathbb{R}\to\mathbb{R}$ satisfy (3)(4) and $\varkappa$ is bounded. If $w(t)$, $t\in[0,T]$ is a Wiener process on $(\Omega,\mathfrak{F},\mathbb{P})$, then the mapping $[0,T]\times C_T\ni(t,w)\mapsto X_t(w)\in\mathbb{R}$ satisfies the equation*

$$c\left(w(t)-\int_0^t\varkappa(X_s(w))\,ds\right)=X_t(w) \tag{5}$$

$\mathbb{P}-$ a.s., iff solves (strongly) Ito's differential equation

$$\begin{aligned} dx(t) &= f(x(t))\,dt+g(x(t))\,dw(t) && (6)\\ x(0) &= c(0) && (7)\end{aligned}$$

$\mathbb{P}-$ a.s.

Proof. Assume that $X_t(w)$ solves (5), and denote

$$\widetilde{w}(t)\triangleq w(t)-\int_o^t\varkappa(X_s(w))$$

From Ito's formula and (3)(4) we get

$$\begin{aligned} & dc(\widetilde{w}(t)) && (8)\\ = & \left[\frac{1}{2}c''(\widetilde{w}(t))-c'(\widetilde{w}(t))\varkappa(X_t(w))\right]dt+c'(\widetilde{w}(t))\,dw(t)\\ = & \left[\frac{1}{2}g(c(\widetilde{w}(t)))g'(c(\widetilde{w}(t)))-g(c(\widetilde{w}(t)))\varkappa(X_t(w))\right]dt+g(c(\widetilde{w}(t)))\,dw(t)\\ = & g(c(\widetilde{w}(t)))\left\{\frac{1}{2}g'(c(\widetilde{w}(t)))-\left[\frac{g'(X_t(w))}{2}-\frac{f(X_t(w))}{g(X_t(w))}\right]\right\}dt\\ & +g(c(\widetilde{w}(t)))\,dw(t)\end{aligned}$$

Since (by the assumption)

$$X_t(w)=c(\widetilde{w}(t))$$

thus the RHS of (8) equals

$$=f(X_t(w))\,dt+g(X_t(w))\,dw(t)$$

Hence $X_t(w)$ solves (6),(7) since $X_0(w)=c(0)$).

Now in the reverse direction. Let $X_t(w)$, $X_0(w)=c(0)$, solves strongly (6),(7). Then

$$\begin{aligned} dX_t(w) &= [f(X_t(w))+g(X_t(w))\varkappa(X_t(w))]\,dt+g(X_t(w))\,[dw(t)-\varkappa(X_t(w))\,dt]\\ &= [f(X_t(w))+g(X_t(w))\varkappa(X_t(w))]\,dt+g(X_t(w))\,d\widetilde{w}(t)\end{aligned}$$

where $\widetilde{w}(t)$ (from Girsanov theorem) is a Wiener process on a "new" space $\left(\Omega, \mathfrak{F}, \widetilde{\mathbb{P}}\right)$ with a measure

$$\begin{aligned}
\widetilde{\mathbb{P}}(A) &= \int_A \Lambda d\mathbb{P}, \;\; A \in \mathfrak{F} \\
\Lambda &= \exp\left[\int_0^T \varkappa\left(X_t(w)\right) dw(t) - \frac{1}{2}\int_0^T \varkappa^2\left(X_t(w)\right) dt\right]
\end{aligned}$$

($\mathbb{E}_{\mathbb{P}}[\Lambda] = 1$, because $\varkappa$ is bounded). It follows that $X_t(w)$ satisfies on $\left(\Omega, \mathfrak{F}, \widetilde{\mathbb{P}}\right)$ the equation

$$\begin{aligned}
& dX_t(w) \\
&= \left[f\left(X_t(w)\right) + g\left(X_t(w)\right)\left[\frac{g'\left(X_t(w)\right)}{2} - \frac{f\left(X_t(w)\right)}{g\left(X_t(w)\right)}\right]\right] dt + g\left(X_t(w)\right) d\widetilde{w}(t) \\
&= g\left(X_t(w)\right)\left[\frac{g'\left(X_t(w)\right)}{2} dt + d\widetilde{w}(t)\right] \qquad (9)
\end{aligned}$$

It can be verified directly, that

$$X_t(w) = c\left(\widetilde{w}(t)\right) \qquad (10)$$

solves (9). Hence, we get

$$X_t(w) = c\left(\widetilde{w}(t)\right) = c\left(w(t) - \int_0^t \varkappa\left(X_s(w)\right) ds\right)$$

on the "old" space $(\Omega, \mathfrak{F}, \mathbb{P})$. ■

Example 4 *(continued). With g and f as above, we have the integral equation, case (a)*

$$X_t(w) = \sinh\left(a + w(t) - \int_0^t \phi\left(X_s(w)\right) ds\right)$$

and Ito's equation

$$dx(t) = \left[\frac{x(t)}{2} - \phi\left(x(t)\right)\sqrt{1 + x^2(t)}\right] dt + \sqrt{1 + x^2(t)} dw(t)$$

case (b)

$$\begin{aligned}
X_t(w) &= \left[sign\left(a + w(t) - \int_0^t \phi\left(X_s(w)\right) ds\right)\right] \\
&\quad \times \left[(1-\alpha)\left|a + w(t) - \int_0^t \phi\left(X_s(w)\right) ds\right|\right]^{1/1-\alpha}
\end{aligned}$$

and Ito's equation

$$dx(t) = \left[\frac{\alpha}{2} sign\left(x(t)\right)\left|x(t)\right|^{2\alpha-1} - \left|x(t)\right|^{\alpha}\phi\left(x(t)\right)\right] dt + \left|x(t)\right|^{\alpha} dw(t)$$

Corollary 5 *Under the conditions of the equivalence theorem, we have*

$$\begin{aligned} d\widetilde{w}(t) &= -\varkappa \circ c(\widetilde{w}(t))\,dt + dw(t) \\ \widetilde{w}(0) &= 0 \end{aligned} \tag{11}$$

Proof. From (5) follows that

$$\begin{aligned} \widetilde{w}(t) &= w(t) - \int_0^t \varkappa(X_s(w))\,ds \\ &= w(t) - \int_0^t \varkappa \circ c\left(w(s) - \int_0^s \varkappa(X_u(w))\,du\right) ds \\ &= w(t) - \int_0^t \varkappa \circ c(\widetilde{w}(s))\,ds \end{aligned}$$

■

Remark 6 *From (10) and (11) we have on* $\left(\Omega, \mathfrak{F}, \widetilde{\mathbb{P}}\right)$

$$X_t\left(\widetilde{w} + \int_0^{\cdot} \varkappa \circ c(\widetilde{w}(s))\,ds\right) = c(\widetilde{w}(t))$$

Example 7 *(a) Since in our example* $\varkappa \circ c(x) = \phi(\sinh(a+x))$, *hence*

$$d\widetilde{w}(t) = -\phi(\sinh(a + \widetilde{w}(t))) + dw(t)$$

(b) here $\varkappa \circ c(x) = \phi\left([sign(a+x)]\,[(1-\alpha)\,|a+x|]^{1/1-\alpha}\right)$, *hence*

$$d\widetilde{w}(t) = -\phi\left([sign(a+\widetilde{w}(t))]\,[(1-\alpha)\,|a+\widetilde{w}(t)|]^{1/1-\alpha}\right) + dw(t)$$

Corollary 8 *(weak solutions) Let* $b(t)$, $t \in [0,T]$ *be a Brownian motions on some probability space* $(\Omega', \mathfrak{F}', \mathbb{P}')$. *Define*

$$\Lambda = \exp\left[-\int_0^T \varkappa \circ c(b(t))\,db(t) - \frac{1}{2}\int_0^T (\varkappa \circ c)^2(b(t))\,dt\right]$$

If

$$\mathbb{E}_{\mathbb{P}}\Lambda = 1$$

then

$$x_b(t) \triangleq c(b(t))$$

is a (weak) solution of (5).

Proof. According to Girsanov theorem

$$\mathbb{P}(A) \triangleq \int_A \Lambda d\mathbb{P}', \quad A \in \mathfrak{F}$$

is a probability measure, $(\Omega, \mathfrak{F}, \mathbb{P})$ is a probability space, and

$$w(t) \triangleq b(t) + \int_0^t \varkappa \circ c(b(s))\, ds$$

is a Wiener process on it. Hence

$$\begin{aligned} x_b(t) &\triangleq c(b(t)) \\ &= c\left(w(t) - \int_0^t \varkappa \circ c(b(s))\, ds\right) \\ &= c\left(w(t) - \int_0^t \varkappa(x_b(s))\, ds\right) \end{aligned}$$

on $(\Omega, \mathfrak{F}, \mathbb{P})$. ■

Remark 9 *$K-$ diffusions starting from random initial conditions can be easily obtained. Let ξ is a random variable on $(\Omega, \mathfrak{F}, \mathbb{P})$, and consider the following generalization of (5)*

$$c\left(\xi + w(t) - \int_0^t \varkappa(X_s(w))\, ds\right) = X_t(w) \tag{12}$$

If ξ is stochastically independent on $w(t)$, $t \in [0, T]$, then the solution of (12) is a $K-$ diffusions with $X_0(w) = c(\xi)$.

4 Applications

4.1 Identification of financial instruments

Consider two financial instruments. Denote their prices by X and Y. Moreover, assume that X and Y are driven by the same Wiener process and assume X is a K$-$ diffusion with c and $\varkappa$ known. How can one identify Y? There is a well known method of a "black box" identification by Norbert Wiener. However, his method is essentially restricted to systems of special kind; input and output must be observable. This is not the case in financial modelling. Here we have the black box $w \to (X_t(w), Y_t(w))$ and one may observe the output only. Hence, this method cannot be applied directly. To overcome this difficulty, observe that, if c^{-1} exists, than the mapping $w \to X_t(w)$ is invertible, and

$$w(t) = c^{-1}(X(t)) - \int_0^t \varkappa(X_s)\, ds$$

is a Wiener process, hence, the input and output of this black box

$$X \to Y_t(X) = Y_t\left(c^{-1}(X(t)) - \int_0^t \varkappa(X_s)\, ds\right)$$

is observable. Now, Wiener's method of nonlinear systems identification can be applied to the $Y-$ black box (see [11], Lecture 10 and 11)

4.2 Fractional diffusions

Let $H \in (0,1)$ be a Hurst index, $B^H(t)$, $t \in [0,T]$ denotes a fractional Brownian motion (fBm) and define

$$\varkappa^H(t,x) = Ht^{2H-1}g'(x) - \frac{f(x)}{g(x)}$$

Proposition 10 *If the mapping $X_t^H(w) : [0,T] \times C([0,T];\mathbb{R}) \to \mathbb{R}$, solves nonlinear integral equation*

$$c\left(B^H(t) - \int_o^t \varkappa^H\left(s, X_s\left(B^H\right)\right) ds\right) = X_t\left(B^H\right)$$

then it solves (strongly) SDE

$$x(t) = c(0) + \int_0^t f(x(s))\, ds + \int_0^t g(x(s))\, dB^H(s)$$

where the stochastic integral is the WIS integral.

Proof. Set

$$w^H(t) = B^H(t) - \int_o^t \varkappa^H\left(s, X_s^H\left(B^H\right)\right) ds$$

From Ito's formula for fBm (p. 161 of [9]) we have

$$\begin{aligned}
& dc\left(w^H(t)\right) \\
= & \left[Ht^{2H-1}c''\left(w^H(t)\right) - \varkappa^H\left(t, X_t^H\left(B^H\right)\right)c'\left(w^H(t)\right)\right] dt + c'\left(w^H(t)\right) dB^H(t) \\
= & \left[Ht^{2H-1}gg'\left(c\left(w^H(t)\right)\right) - \left[Ht^{2H-1}g'\left(X_t^H\left(B^H\right)\right) - \frac{f\left(X_t^H\left(B^H\right)\right)}{g\left(X_t^H\left(B^H\right)\right)}\right] g\left(c\left(w^H(t)\right)\right)\right] dt \\
& + g\left(c\left(w^H(t)\right)\right) dB^H(t) \\
= & f\left(X_t^H\left(B^H\right)\right) dt + g\left(X_t^H\left(B^H\right)\right) dB^H(t)
\end{aligned}$$

since by the assumption $c\left(w^H(t)\right) = X_t^H\left(B^H\right)$. ■

4.3 Smooth densities

Set $\widetilde{F}(x) = \mathbb{P}(\widetilde{w}(t) < x)$. Then

$$\begin{aligned} \mathbb{P}(X_t(w) < x) &= \mathbb{P}(c(\widetilde{w}(t)) < x) \\ &= \widetilde{F} \circ c^{-1}(x) \end{aligned}$$

Hence, the smoothness density problem for $X_t(w)$, is reduced to investigation of ordinary function $\widetilde{F} \circ c^{-1}$.

Example 11

$$\begin{aligned} &(a)\ \ \widetilde{F} \circ c^{-1}(x) = \widetilde{F}\left(\sinh^{-1}(a+x)\right) \\ &(b)\ \ \widetilde{F} \circ c^{-1}(x) = \widetilde{F}\left(sign\,(a+x)\,(1-\alpha)^{1-\alpha}\,|a+x|^{1-\alpha}\right) \end{aligned}$$

Acknowledgement 12 *I would like to thank Professor Moshe Zakai for his remarks and suggestions.*

References

[1] A.N. Kolmogorov, Uber die analytischen methoden in der wahrscheinlichkeitsrechnung, Math. Ann. 104, (1931), 415-458

[2] K. Ito, On stochastic differential equations, Memoirs Amer. Math. Soc. 4, (1951), 1- 51

[3] D.W. Stroock, S.R.S. Varadhan, Multidimensional Diffusion Processes, Springer-Verlag 1979

[4] E. Isobe, S. Sato, Wiener-Hermite expansion of a process generated by an Ito stochastic differential equation, J. Appl. Prob. 20 (1983), 754-765

[5] K. Ito, Selected Papers, Edited by D.W. Stroock and S.R.S. Varadhan, Springer-Verlag 1987

[6] D.W. Stroock, Markov Processes from K. Ito's Perspective, Princeton Univ. Press 2003

[7] H. Sussman, An interpretation of stochastic differential equations as orinary differential equations which depend on the sample point, Bull.Amer. Math. Soc. 83, (1977) 296-298

[8] H. Doss, Liens entre equations differentielles stochastiques et ordinaires, Ann. Inst. Henri Poincare 13, (1977), 99-125 F.

[9] F. Biagini, Y. Hu, B. Oksendal, T. Zhang, Stochastic Calculus for Fractional Brownian Motion and Applications, Springer, 2008

[10] I. Karatzas, S.E. Shreve, Brownian Motion and Stochastic Calculus, Springer-Verlag, 1991

[11] N. Wiener, Nonlinear Problems in Random Theory, Tech. Press of MIT, John Wiley&Sons, Chapman&Hall, 1958